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Plate-injection into a separated supersonic boundary layer. Part 2. The transition regions

By K. STEWARTSON

Department of Aeronautical and Astronautical Engineering,
The Ohio State University, Columbus†

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The model proposed by Smith & Stewartson (1973), to describe the separated boundary layer induced by strong injection over a finite length of a flat plate in a supersonic mainstream, is shown to provide the basis for a fully consistent solution of the Navier–Stokes equations for this problem, valid in the limit of infinite Reynolds number. The solution takes the form of asymptotic expansions in each of a large number of overlapping regions of the flow field, which are consistently matched across areas of common validity.

1. Introduction

Apart from its practical value, as a means of effecting a reduction in heat transfer or of blowing off the boundary layer, the injection problem of injecting fluid across a permeable wall has considerable mathematical interest too. It seems to provide the simplest example of a flow field involving separation for which an asymptotic solution of the Navier–Stokes equations uniformly valid in the limit of infinite Reynolds number can be written down. In an earlier paper (Smith & Stewartson 1973) with the same principal title, and which we shall subsequently refer to as P, a suggested model for the flow field was described and is shown schematically in figure 1. Here the wall is a fixed flat plate occupying the part $0 < x^* < L$ of the x^* axis of a Cartesian co-ordinate system Ox^*y^*. At a large distance upstream the fluid is assumed to be in uniform motion with velocity U^*_∞ parallel to Ox^* and with Mach number M_∞ satisfying the condition $M_\infty > 1$ for a supersonic flow. The suffixes ∞ and w are used to indicate conditions far upstream and on the plate respectively. For example, the plate is assumed to be maintained at a constant temperature T^*_w. Fluid is injected into the region $y^* > 0$ above the plate, from the part $0 < x^*_0 \leqslant x^* \leqslant x^*_1 < L$, with uniform velocity V^*_w and at a constant density $\rho^*_w = \rho^*_\infty T^*_\infty/T^*_w$, i.e. at the ambient density (and also pressure, as we shall see, for $V^*_w \ll U^*_\infty$). A laminar boundary layer is formed in the vicinity of O and is assumed to separate at the point $(x^*_s, 0)$, where $x^*_s < x^*_0$, thereafter forming a detached shear layer. We define the Reynolds number as

$$Re = U^*_\infty x^*_s/\nu^*_\infty = \epsilon^{-8}, \tag{1.1}$$

where ν^* is the kinematic viscosity, and assume that $0 < \epsilon \ll 1$. With the further assumption that $V^*_w = O(\epsilon^3 U^*_\infty)$ we claim that the solution of the full Navier–

† Permanent address: Department of Mathematics, University College, London.

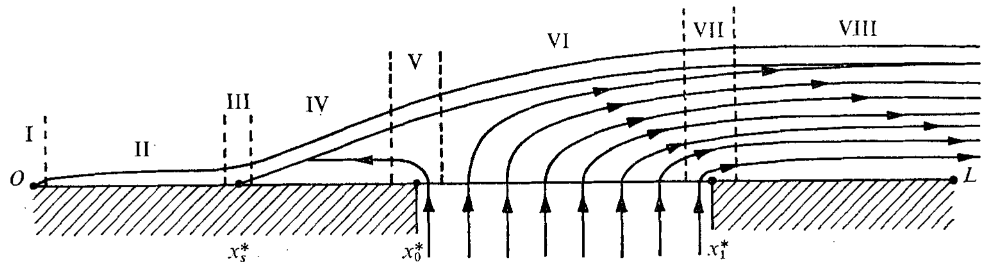

FIGURE 1. The various regions of the flow field.

Stokes equations, subject to the appropriate boundary conditions for this problem, can be regarded as a combination of solutions in a (large) number of overlapping regions, each of which may be written down as a series of ascending powers of ϵ (and possibly also of $\log \epsilon$ eventually) whose coefficients are functions of certain appropriately scaled spatial variables. Further, these functions are found by the solution of partial differential equations and satisfying appropriate boundary conditions in which the parameter ϵ is explicitly absent.

Proceeding downstream, the first of these regions (I) is the neighbourhood of O, where $|x^*|, |y^*| = O(x_s^* \epsilon^4)$, and the fundamental problem here is the solution of the full Navier–Stokes equations for a semi-infinite flat plate. Such a solution is not yet available, but the corresponding solution for an incompressible fluid has been computed by Van de Vooren & Dijkstra (1970). A substantially greater computational effort is needed to extend their study to a compressible fluid. In addition the solution is probably not unique when $M_\infty > 1$ without some downstream restriction on the pressure in view of the possibility of self-induced separation (Stewartson & Williams 1969). Otherwise there does not appear to be any difficulty in principle in carrying out the extension.

The second region (II) is the boundary layer ahead of separation ($0 < x^* < x_s^*$) and is well understood.

The third region (III) is the neighbourhood of separation and here a systematic expansion procedure has been worked out (Neiland 1969; Stewartson & Williams 1969; Stewartson 1974) based on the Lighthill (1953) model and involving a triple-deck structure. It extends for a distance $O(\epsilon^3 x_s^*)$ upstream and downstream of x_s^* and is subdivided laterally into three decks of thicknesses $\epsilon^5 x_s^*$, $\epsilon^4 x_s^*$ and $\epsilon^3 x_s^*$. The lower deck, of thickness $\epsilon^5 x_s^*$, is the only one in which extensive numerical computation is required, the others involving no more than quadratures. It is found to be capable of separating spontaneously, provided only that the associated pressure rise is permitted by the flow properties further downstream of III. Once the position of separation x_s^* is fixed, the solution of the equation of the lower deck upstream of separation is unique, the mathematical problem being well posed. Downstream of separation, however, the uniqueness is lost without an extra condition, on that part of the velocity profile which is reversed in direction, at some station of $x^* > x_s^*$ since small disturbances can propagate in the direction of x^* decreasing through the reversed flow at the bottom of the lower deck. In the initial attack on the problem it was decided, intuitively, to look for solutions in which the relative magnitude of the reversed velocity becomes vanishingly small as $(x^* - x_s^*)/\epsilon^3 x_s^* \to \infty$; when P was written sufficient numerical evidence was available to suggest to the authors that the hypothesis is reasonable. Subsequently Stewartson & Williams (1973) were able to set up a completely consistent asymptotic expansion for the solution in the lower deck, valid, on this hypothesis, in the limit $(x^* - x_s^*)/\epsilon^3 x_s^* \to \infty$, and in § 2 below we shall describe its main properties. Later Williams (1974) refined his numerical procedure and was able to make a favourable comparison with this expansion. It is claimed that a complete description of the separation region is now available, subject of course to the hypothesis mentioned above. Through the courtesy of Professor A. F. Messiter, the author has recently learned that the leading term in the asymptotic expansion had been worked out at an earlier date, by Nieland (1971).

The next region (IV) is the plateau in which the original boundary layer detaches from the wall, becoming a free shear layer inclined to it at a constant angle $O(\epsilon^2)$ and separating the oncoming inviscid supersonic flow above from the slow reversed flow below. The pressure is almost constant, following the rise $O(\epsilon^2 p_\infty^*)$ through the triple deck at separation, hence the name plateau for IV, but there is a weak positive pressure gradient $O(\epsilon^4 p_\infty^*/x_s^*)$ which serves to drive the fluid near the wall in the direction of x^* decreasing and towards x_s^*. The flow properties in IV are studied in §§ 3 and 5.

In order to compute the reversed velocity in IV an initial profile is required at the downstream end of the region and the associated mass flux is provided by a short length of the permeable part of the plate. A fifth region (V) is therefore needed in the neighbourhood of x_0^*, in which some of the injected fluid is turned in the direction of x^* decreasing. We shall study its properties in § 4 and establish that its streamwise extent is $O(\epsilon L)$. It is noted that the injected fluid which moves upstream into the reversed-flow part of IV is eventually entrained into the detached shear layer and swept downstream.

The next region (VI) comprises the majority of the injection region of the plate $(x_0^* < x^* < x_1^*)$ and has been studied in P. The further details about the flow properties provided in this paper do not affect the solutions given there, to leading order, and there does not appear to be any difficulty in principle about taking them into account in the higher-order terms of the asymptotic expansion of the solution in powers of ϵ.

A critical assumption made in the discussion of the solution in VI is that the pressure at x_1^* is equal to p_∞^*, if terms $O(\epsilon^3 p_\infty^*)$ are neglected. This means that, at $x^* = x_1^*$, the pressure gradient is discontinuous and so another transition region (VII) is needed to smooth it out. This region also straightens out the streamlines in the lower part of the boundary layer and gives birth to a sub-boundary layer of thickness $O(\epsilon^{11/3} x_s^*)$. The properties of region VII are discussed in § 6.

The remaining regions downstream of x_1^* have much in common with those occurring in the study of the boundary layer on an impermeable finite flat plate (trailing edge, a rear wake, far wake, etc.), and their properties follow in a similar way. A detailed discussion does not seem called for here and the reader is referred to Stewartson (1974) for further information.

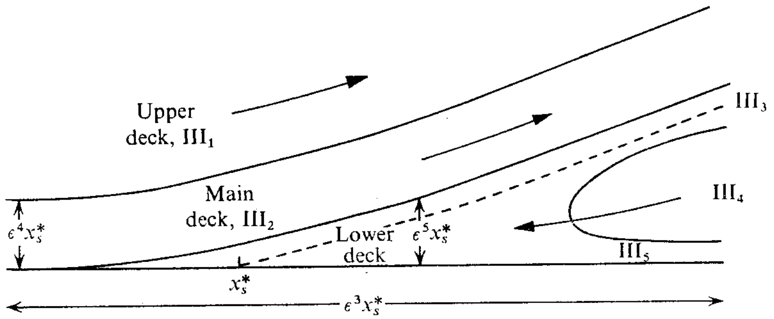

FIGURE 2. The subregions of III (the triple deck).

2. The terminal structure of the triple deck at separation

The streamwise extent and total width of the triple deck are both $O(\epsilon^3 x_s^*)$ and it divides laterally into three decks of which the upper deck consists of fluid originally ($x^* < x_s^*$) outside the boundary layer. In this deck the velocity perturbation from the original uniform state is $O(\epsilon^2 U_\infty^*)$, and since its width is $O(\epsilon^3 x_s^*)$, the governing equations reduce to the Prandtl–Glauert equation of linearized inviscid supersonic flow. The main deck, of thickness $O(\epsilon^4 x_s^*)$, lies below the upper deck, occupying a relatively small part of the triple-deck region, and the fluid moving through it came originally from the boundary layer. This fluid behaves passively, being merely lifted up by the expansion of the lower deck below it, and the induced pressure gradient is relatively insignificant. The lower deck, of thickness $O(\epsilon^5 x_s^*)$, lies below the main deck and in turn occupies a small part of the triple deck, even relative to the main deck. The fluid motion in the lower deck is controlled by the incompressible boundary-layer equations, but with novel boundary conditions. Of these the most interesting is that the pressure gradient is linked to overall properties of the solution through the interaction between the three decks. One immediate consequence is that a singularity in the solution at separation is impossible and there is instead a smooth evolution of the velocity profile from a uniform shear into one which is partly reversed.

When $(x^* - x_s^*)/\epsilon^3 x_s^*$ is large and positive but $(x^* - x_s^*)/x_s^*$ is small, i.e. at the downstream limit of validity of the triple deck, region III, it subdivides further into five subregions as shown in figure 2. From the asymptotic studies of Neiland (1971) and Stewartson & Williams (1973), and using the numerical data of Williams (1974), we may describe the flow properties as follows. Let (u^*, v^*) be the velocity components along the (x^*, y^*) directions respectively. Then, working down towards the plate, we have, successively, the following subregions.

Subregion III_1 (*the upper deck*)

This is defined by $y^* = O(x^* - x_s^*)$ and the flow properties are

$$\left.\begin{aligned} u^*/U_\infty^* &= 1 - [\alpha_0 \epsilon^2/(M_\infty^2 - 1)^{\frac{1}{2}}] + O(\epsilon^3), \\ v^*/U_\infty^* &= \epsilon^2 \alpha_0 + O(\epsilon^3), \\ p^* &= p_\infty^* + [\alpha_0 \epsilon^2 \rho_\infty^* U_\infty^{*2}/(M_\infty^2 - 1)^{\frac{1}{2}}] + O(\epsilon^3), \end{aligned}\right\} \tag{2.1}$$

where $$\alpha_0 = \lambda^{\frac{1}{2}} C^{\frac{1}{4}} (M_\infty^2 - 1)^{\frac{1}{4}} P_0. \tag{2.2}$$

We note at this point that the fluid is assumed to satisfy Chapman's viscosity law and that C is Chapman's constant, the Prandtl number is taken to be unity, $\lambda = 0{\cdot}3321$, and P_0 is a numerical constant the best estimate for which is 1·79 (Williams 1974). Previous estimates (e.g. in P) gave $P_0 = 1{\cdot}8$. Thus, at the downstream end of the triple deck, the fluid in III is moving with uniform velocity in a direction making an angle $\alpha_0 \epsilon^2$ with the plate ($y^* = 0$).

Subregion III_2 *(the main deck)*

This is defined by $\bar{y}_3 = O(1)$, where $\bar{y}_3 = y^*/\epsilon^4 x_s^*$, and the flow properties are

$$\left. \begin{aligned} u^*/U_\infty^* &= U_2(\bar{y}_3) - \epsilon U_2'(\bar{y}_3)\, \alpha_0 (x^* - x_s^*)/\epsilon^3 x_s^* + \ldots, \\ v^*/U_\infty^* &= \epsilon^2 U_2(\bar{y}_3)\, \alpha_0 + O(\epsilon^3), \end{aligned} \right\} \tag{2.3}$$

with p^* as in (2.1). Here U_2 is the undisturbed velocity profile in the boundary layer just upstream of the triple deck. A more revealing form of u^* in III_2 is

$$u^* = U_\infty^* U_2(y_3) + O(\epsilon^2),$$

where $$y_3 = [y^* - \epsilon^2 \alpha_0 (x^* - x_s^*)]/\epsilon^4 x_s^*, \tag{2.4}$$

which shows that the principal change is not in the velocity profile, but in the direction of the streamlines in the main deck, now seen to be inclined to the plate at an angle $\alpha_0 \epsilon^2$, just as in III_1.

Subregion III_3 *(top part of the lower deck)*

This is defined by $y_3 = O[\{(x^* - x_s^*)/x_s^*\}^{\frac{1}{3}}]$ and the flow properties are

$$u^* = \epsilon U_\infty^* \lambda^{\frac{2}{3}} x_3^{\frac{1}{3}} G_0'(\xi_3), \tag{2.5}$$

where $$\xi_3 = \frac{\lambda^{\frac{1}{3}} y_3}{\epsilon C^{\frac{1}{2}} x_3^{\frac{1}{3}}} \frac{T_\infty^*}{T_w^*}, \quad x_3 = \frac{x^* - x_s^*}{\epsilon^3 x_s^*}$$

and $$G_0''' + \tfrac{2}{3} G_0 G_0'' - \tfrac{1}{3} G_0'^2 = 0 \tag{2.6}$$

with $$G_0'' \to 1 \quad \text{as} \quad \xi_3 \to \infty, \quad G_0' \to 0 \quad \text{as} \quad \xi_3 \to -\infty.$$

There is an equivalent form for v^*. From the forms taken by u^* in III_2 and III_3, it is clear that, as x^* increases further, they may be combined into one region in which the original boundary layer (in II) becomes a free shear layer bounded on one side by a uniform stream and on the other by fluid which is virtually at rest.

Subregion III_4 *(central part of lower deck)*

This is defined by $0 < y^*/\epsilon^2 \alpha_0 (x^* - x_s^*) < 1$ and the fluid velocity is reversed in direction, being given by

$$\frac{u^*}{U_\infty^*} = -\frac{\epsilon C_0}{x_3^{\frac{1}{3}} P_0} \frac{\lambda^{-\frac{1}{6}} C^{\frac{1}{4}} T_w^*}{(M_\infty^2 - 1)^{\frac{1}{4}} T_\infty^*} + O(\epsilon^2), \tag{2.7}$$

where $C_0 = -G_0(-\infty) = 1{\cdot}252$. The fluid density in III_4 is the same as the density ρ_w^* at the wall, so that v^* follows from the equation of continuity for an incom-

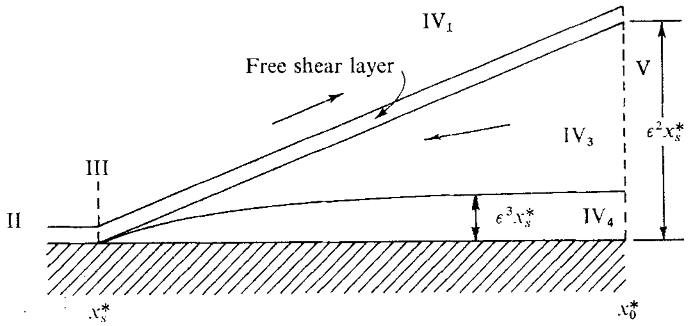

FIGURE 3. The subregions of IV (the plateau).

pressible fluid. The pressure variation follows from the x^* component of the momentum equation and we obtain

$$p^* = p_\infty^* + \frac{\alpha_0 \epsilon^2 \rho_\infty^* U_\infty^{*2}}{(M_\infty^2 - 1)^{\frac{1}{2}}} - \frac{\epsilon^2 C_0^2 \lambda^{-\frac{1}{3}} C^{\frac{1}{2}} \rho_\infty^* U_\infty^{*2} T_w^*}{2P_0^2 x_3^{\frac{2}{3}} T_\infty^* (M_\infty^2 - 1)^{\frac{1}{2}}}. \tag{2.8}$$

This equation also holds in the other subregions and shows how the limiting form in (2.1) is approached as $x_3 \to \infty$. A final point about III$_4$ is that viscous forces are not significant.

Subregion III$_5$ (*bottom of lower deck*)

This region is defined by $y^* = O(\epsilon^2 (x^* - x_s^*)^{\frac{2}{3}} x_s^{*\frac{1}{3}})$ and viscous forces are important here since the slip velocity implied by (2.7) must be reduced to zero at the plate. The appropriate structure is defined by the self-similar solution of the incompressible boundary-layer equations corresponding to a mainstream velocity (2.7).

3. The structure of region IV. Part 1

We postulate that region IV, comprising that part of the plate between the triple deck at x_s^* and a transition region near x_0^*, the point at which injection begins, may itself be subdivided into four parts as shown in figure 3.

(*a*) Subregion IV$_1$, which is a continuation of III$_1$ and has the same properties, namely that the fluid is in uniform motion inclined at an angle $\alpha_0 \epsilon^2$ to the plate and the velocity components and pressure are given by (2.1).

(*b*) Subregion IV$_2$, which is a shear layer of thickness $O(\epsilon^4 x_s^*)$ in the neighbourhood of the straight line $y^* = \alpha_0 \epsilon^2 (x^* - x_s^*)$.

(*c*) Subregion IV$_3$, which occupies the majority of the area between IV$_2$ and the plate and in which the x^* component of velocity is $O(\epsilon^2 U_\infty^*)$ and directed in the opposite direction from that in IV$_1$. Viscous forces are negligible here.

(*d*) Subregion IV$_4$, which is a viscous sub-boundary layer of thickness $O(\epsilon^3 x_s^*)$ adjacent to the plate, whose purpose is to remove the slip velocity at the bottom of IV$_3$.

We begin a more detailed account of the properties of these subregions with IV$_2$. Since the shear layer here is $O(\epsilon^4 x_s^*)$ in width, while it extends downstream a distance $O(L)$ and remains close to the line $0 = y_4^* \equiv \epsilon^4 x_s^* y_3$, the evolution of the shear layer must be determined by the solution of the boundary-layer equations

$$\left.\begin{aligned} \rho^* u^* \frac{\partial u^*}{\partial x^*} + \rho^* v_4^* \frac{\partial u^*}{\partial y_4^*} &= \frac{\partial}{\partial y_4^*}\left(\mu^* \frac{\partial u^*}{\partial y_4^*}\right), \\ \frac{\partial}{\partial x^*}(\rho^* u^*) + \frac{\partial}{\partial y_4^*}(\rho^* v_4^*) &= 0, \end{aligned}\right\} \tag{3.1}$$

where
$$v_4^* = v^* - \epsilon^2 \alpha_0 u^*,$$

together with a similar equation for T^*. The associated boundary conditions are

$$\left.\begin{aligned} u^* \to U_\infty^*, 0, \quad T^* \to T_\infty^*, T_w^* \quad &\text{as} \quad y_4^* \to \pm\infty, \\ u^* = U_\infty^* U_2(y_3), \quad T^* = T_\infty^* T_2(y_3) \quad &\text{at} \quad x^* = x_s^*, \end{aligned}\right\} \tag{3.2}$$

where $U_\infty^* U_2$ and $T_\infty^* T_2$ are the velocity and temperature profiles in the boundary layer at the downstream end ($x^* = x_s^*$) of region II. The last conditions of (3.2) are expressions of the need for the forms of u^* and T^* in IV_2 to be continuous with those in II. The flow properties in III_2 make this need evident. Equations (3.1) simplify on writing

$$\epsilon^4 x_s^* y_4 = C^{-\frac{1}{2}} \int_0^{y_4^*} \frac{\rho^* \, dy_4^*}{\rho_\infty^*}, \quad u^* = U_\infty^* \frac{\partial \psi_4}{\partial y_4}, \quad x^* - x_s^* = x_s^* x_4, \tag{3.3}$$

and using the Chapman viscosity law, when they reduce to a single equation for ψ_4, independent of T^*, namely

$$\frac{\partial \psi_4}{\partial y_4} \frac{\partial^2 \psi_4}{\partial x_4 \partial y_4} - \frac{\partial \psi_4}{\partial x_4} \frac{\partial^2 \psi_4}{\partial y_4^2} = \frac{\partial^3 \psi_4}{\partial y_4^3}, \tag{3.4}$$

together with the boundary conditions

$$\partial \psi_4 / \partial y_4 \to 1, 0 \quad \text{as} \quad y_4 \to \pm\infty, \quad x_4 > 0, \tag{3.5}$$

and
$$\psi_4 = \int_0^{y_4} U_2(y_4) \, dy_4 \quad \text{at} \quad x_4 = 0. \tag{3.6}$$

The equation for T^* is similar to (3.4) but we do not need it here.

The solution of (3.4), subject to (3.5) and (3.6), was kindly obtained for the author by Mr P. G. Daniels using a method devised by Smith (1974). The solution is not unique because an arbitrary function of x_4 may be added to y_4, without disturbing either the equations or the boundary conditions, provided only that it vanishes at $x_4 = 0$. This is as it should be for the position of the shear layer is intrinsically indeterminate to $O(\epsilon^4 x_s^*)$ and is fixed by the weak pressure gradient which drives the fluid in IV_3. When $0 < x_4 \ll 1$ the solution can be obtained as a double-structured expansion, parallel to that worked out by Goldstein (1930) for the near-wake velocity field behind a finite flat plate. The main difference is the change in boundary conditions from $\partial \psi_4 / \partial y_4 \to 0$ as $y_4 \to -\infty$ to

$$\psi_4 = \partial^2 \psi_4 / \partial y_4^2 = 0 \quad \text{at} \quad y_4 = 0,$$

but the formal expansions in each of the two subregions are the same. The leading terms in the present problem are identical with the leading terms of

x_4	v_E	x_4	v_E
0·000250	9·001	1·9046	0·369
0·00675	2·966	7·605	0·205
0·03125	1·773	13·05	0·160
0·11864	1·121	25·13	0·117
0·5001	0·660	49·01	0·085
0·9861	0·500		

TABLE 1

the expansions in III_2 and III_3, due allowance being made for the different scaling laws adopted, showing that there is a consistent match so far with region III.

The most important feature of the solution is the entrainment velocity

$$v_E(x_4) = -\lim_{y_4 \to -\infty} (\partial \psi_4 / \partial x_4), \tag{3.7}$$

which *is* uniquely defined and tells us how much fluid is needed from IV_3 to feed the shear layer. A set of representative values of v_E derived from Mr Daniel's computations is displayed in table 1.

In physical terms the entrainment velocity is

$$\epsilon^4 U^*_\infty C^{\frac{1}{2}} \frac{T^*_w}{T^*_\infty} v_E\{(x^* - x^*_s)/x^*_s\} \tag{3.8}$$

and for a match with IV_3 it is essential that, in IV_3, v^* should take on this value when $y^* = \alpha_0 \epsilon^2 (x^* - x^*_s)$. At large values of x_4 the solution of (3.4) takes on the self-similar form described by Chapman (1950) in his discussion of mixing layers and $v_E x_4^{\frac{1}{2}}$ tends to the limit 0·619 as $x_4 \to \infty$. A useful approximate formula for v_E, valid when x_4 is either large or small and in error by about 5 % at $x_4 = 1$, is

$$v_E \simeq 0{\cdot}619 x_4^{-\frac{1}{3}} (x_4 + 1{\cdot}150)^{-\frac{1}{6}}. \tag{3.9}$$

It is noted that this shear layer continues beyond the end of IV into regions V–VIII, where it is finally parallel to the plate again. With an appropriate modification to the definition of y^*_4, the governing equations and boundary conditions are still valid in these regions. In accordance with Chapman's mixing-layer theory its width is proportional to $x_4^{\frac{1}{2}}$ when $x_4 \gg 1$ and hence for the plate to modify its structure it is necessary that x^*/x^*_s be large.

Next we consider IV_3. This subregion is bounded above by the straight line $y^*_4 = 0$, i.e. $y^* = \epsilon^2 \alpha_0 (x^* - x^*_s)$, below by the straight line $y^* = 0$, and extends as far as $x^* = x^*_0$. Transition regions occur in the neighbourhood of $y^* = 0$ and of $x^* = x^*_0$, which we shall discuss in §§ 4 and 5. The shape of the boundary to IV_3 suggests that the appropriate length scales are $O(x^*_s)$ and $O(\epsilon^2 x^*_s)$ in the x^* and y^* directions respectively. Further, the entrainment velocity required by IV_2 suggests that $v^* = O(\epsilon^4 U^*_\infty)$ and this in turn implies that $u^* = O(\epsilon^2 U^*_\infty)$. These velocity components are also of the right order of magnitude to match with the velocity components in III_4. Thus from (2.7), $u^* = O(\epsilon^2 U^*_\infty)$ when

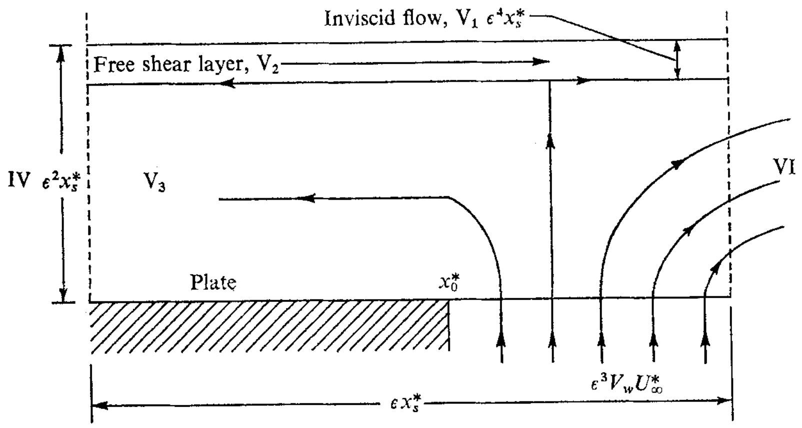

FIGURE 4. The subregions of V (the onset of blowing).

$\epsilon^3 x_3 \equiv x_4 = O(1)$. In addition, ρ and T are constant in IV_3 and equal to their values at the plate since $|u^*/U_\infty^*| \ll 1$.

Write

$$\left.\begin{aligned} x^* - x_s^* &= x_s^* x_4, \quad y^* = \epsilon^2 x_s^* Y_4, \\ u^* &= \epsilon^2 U_\infty^* u_4(x_4, Y_4), \quad v^* = \epsilon^4 U_\infty^* v_4, \\ p^* - p_\infty^* &= [(\rho_\infty^* U_\infty^{*2} \alpha_0 \epsilon^2)/(M_\infty^2 - 1)^{\frac{1}{2}}] + \rho_w^* U_\infty^{*2} \epsilon^4 P_4(x_4, Y_4). \end{aligned}\right\} \tag{3.10}$$

Then on substituting into the full equations of motion and letting $\epsilon \rightarrow 0$ we obtain

$$\left.\begin{aligned} -\frac{\partial P_4}{\partial x_4} = u_4 \frac{\partial u_4}{\partial x_4} + v_4 \frac{\partial u_4}{\partial Y_4}, \quad \frac{\partial P_4}{\partial Y_4} = 0, \\ (\partial u_4/\partial x_4) + (\partial v_4/\partial Y_4) = 0, \quad \rho^* = \rho_w^*, \quad T^* = T_w^*. \end{aligned}\right\} \tag{3.11}$$

The boundary conditions are

$$v_4 = \begin{cases} 0 & \text{when} \quad Y_4 = 0, \quad x_4 > 0; \\ C^{\frac{1}{2}} v_E(x_4)\, T_w^*/T_\infty^* & \text{when} \quad Y_4 = \alpha_0 x_4; \end{cases}$$

in addition we need an initial condition on u_4 at the junction with the transition region at the beginning of the permeable part of the plate, since $u_4 < 0$ in IV_3. Specifically if $x_0^* - x_s^* = \delta_0 x_s^*/\alpha_0$, where δ_0 is a constant, assumed to be of order one but determined by the overall injection into the boundary layer, we need to know the value $U_5(Y_4)$ of u_4 when $x_4 = \delta_0/\alpha_0$ and $0 < Y_4 < \delta_0$. We shall derive a formula for U_5 in the next section and return to the discussion of IV afterwards.

4. The structure of region V

The various subregions of V are shown in figure 4. Of these, V_1, being the continuation of the inviscid subregion IV_1, and V_2, being the continuation of the shear layer IV_2, require little discussion since their properties hardly change and their effect on the subregion below is relatively small. Their boundaries appear to be parallel to the plate in figure 4 because of the spatial scaling adopted.

The subregion V_3 is of significance for the determination of the value of $U_5(Y_4)$, the missing boundary condition needed to complete the specification of IV. Fluid is injected at a uniform velocity $\epsilon^3 V_w U^*_\infty$ into this subregion downstream from $x^* = x^*_0$ and some of this fluid must travel upstream into IV_3 and from there be entrained into IV_2 and swept downstream, since none can penetrate beyond x^*_s. It follows that the fluid injected over a length $O(\epsilon x^*_s)$ moves upstream and this suggests that the transition region is $O(\epsilon x^*_s)$ in length. Hence since $v^* = O(\epsilon^3 U^*_\infty)$ at the wall in $x^* > x^*_0$, $u^* = O(\epsilon^2 U^*_\infty)$ in V_3 and so we write

$$\left.\begin{aligned} x^* &= x^*_0 + \epsilon x^*_s x_5, \qquad y^* = \epsilon^2 x^*_s Y_5, \\ u^* &= \epsilon^2 U^*_\infty u_5(x_5, Y_5), \qquad v^* = \epsilon^3 U^*_\infty v_5(x_5, Y_5), \\ (p^* - p^*_\infty)/U^{*2}_\infty &= (\epsilon^2 \alpha_0 \rho^*_\infty/(M^2_\infty - 1)^{\frac{1}{2}}) + \rho^*_w \epsilon^4 p_5(x_5, Y_5), \end{aligned}\right\} \tag{4.1}$$

and take ρ and T constant and equal to their values at the plate. The governing equations are then the same as (3.11) with the suffix 5 replacing 4 and boundary conditions are

$$v_5 = \left\{\begin{array}{lll} 0, V_w & \text{at} \quad Y_5 = 0, & x_5 \lessgtr 0, \\ 0 & \text{at} \quad Y_5 = \delta_0, & \text{where} \quad x^*_s \delta_0 = \alpha_0(x^*_0 - x^*_s). \end{array}\right\} \tag{4.2}$$

The reader is reminded that while x^*_0 is prescribed as the point on the plate where blowing begins, x^*_s is one of the parameters of the problem that has to be determined, and depends both on the rate of injection and on the length of blowing. A first approximation to its value, sufficient for the present purposes, is given in P and so we may now take δ_0 as given. In order to complete the specification of the solutions in V_3 we need the mass flux of fluid into IV_3 and this is

$$\epsilon^4 x^*_s U^*_\infty \rho^*_w C^{\frac{1}{2}} \frac{T^*_w}{T^*_\infty} \int_0^{\delta_0/\alpha_0} v_E(x_4)\, dx_4. \tag{4.3}$$

Hence the length $\bar{x}_5$ of blowing required to supply this mass flux is given by

$$\bar{x}_5 = V_w^{-1} C^{\frac{1}{2}} \frac{T^*_w}{T^*_\infty} \int_0^{\delta_0/\alpha_0} v_E(x_4)\, dx_4. \tag{4.4}$$

The solution now follows familiar lines. If Ψ_5 is the stream function of the flow field in V_3, so that $\Psi_5(x_5, 0) = 0$ if $x_5 < 0$ and $\Psi_5(x_5, 0) = -V_w x_5$ if $x_5 > 0$, then

$$\tfrac{1}{2} u_5^2 = -P_5(x_5) + f(\Psi_5) \tag{4.5}$$

and the arbitrary function f is determined by the condition that $u_5 = 0$ when $Y_5 = 0$. Hence $f(\Psi_5) = P_5(t_5)$, $(t_5, 0)$ being the point where the streamline through (x_5, Y_5) emerges from the plate. We then have

$$\delta_0 = \int_{-V_w x_5}^{-V_w \bar{x}_5} \frac{d\Psi_5}{\{2[P_5(t_5) - P_5(x_5)]\}^{\frac{1}{2}}} \quad (\Psi_5 = -V_w t_5), \tag{4.6}$$

when $x_5 > \bar{x}_5$, since the fluid is then moving in the direction of x_5 increasing. The solution of this integral equation is

$$P_5 = -(\pi^2 V_w^2/8\delta_0^2)(x_5 - \bar{x}_5)^2 \tag{4.7}$$

apart from an arbitrary constant which cannot be determined at present. The expansion in powers of ϵ would have to be carried further in selected subregions of the flow field before a value could be assigned to this constant. When $x_5 \gg 1$, equation (4.7) matches with the solution given in P for region VI when $(x^* - x_0^*)/x_s^* \ll 1$.

On the other hand, when $x_5 < \bar{x}_5$ the corresponding formula for δ_0 is

$$\delta_0 = -\int_{\Psi_0}^{-V_w \bar{x}_5} \frac{d\Psi_5}{\{2[P_5(t_5) - P_5(x_5)]\}^{\frac{1}{2}}} \quad (\Psi_5 = -V_w t_5), \tag{4.8}$$

where $\Psi_0 = -V_5 x_5$ if $x_5 > 0$ and $\Psi_0 = 0$ if $x_5 < 0$. The lower limit of integration expresses the fact that no fluid injected into V_3 if $x_5 < 0$. The negative sign precedes the integral in (4.8) because $u_5 < 0$, the injected fluid moving upstream towards IV_3. The solution of (4.8) is the same as (4.7) if $x_5 > 0$, but if $x_5 < 0$, P_5 is constant and equal to its value at $x_5 = 0$, according to (4.7).

The corresponding values of u_5 are

$$u_5 = \begin{cases} \dfrac{\pi V_w}{2\delta_0}(x_5 - \bar{x}_5)\sin\dfrac{\pi Y_5}{2\delta_0} & \text{if} \quad x_5 > 0, \quad (4.9) \\ -\dfrac{\pi V_w \bar{x}_5}{2\delta_0}\sin\dfrac{\pi Y_5}{2\delta_0} & \text{if} \quad x_5 < 0, \quad (4.10) \end{cases}$$

which gives us $U_5(Y_4)$, the initial condition on u_4 needed to complete the specification of IV_3, Y_4 being equal to Y_5 at matching.

It is noted that there is a discontinuity in v_5 and in the pressure gradient at $x_5 = 0$ which necessitate another interior layer of length $O(\epsilon^2 x_s^*)$ to smooth them out. The elucidation of the structure of this layer seems to follow similar lines to those of subregion VII_3 discussed in § 6 below and does not need further comment. Since $u_5 = 0$ at $Y_5 = 0$ for all x_5 no boundary layer is needed at this stage, but on continuing the expansion a slip velocity appears at the next stage. The ensuing boundary layer has a structure somewhat similar to that of VII_4 below. Finally, within a distance $O(\epsilon^4 x_s^*)$ of the point $(x_0^*, 0)$ all these expansion schemes must fail and a central region be considered where the full Navier–Stokes equations need to be solved.

5. The structure of region IV. Part 2

Having obtained the value of u_4 at $x_4 = \delta_0/\alpha_0$ in the previous section, we may now complete the discussion of the flow properties in region IV. We define the stream function $\Psi_4(x_4, Y_4)$ in IV_3 from the equation of continuity in (3.11) and take $\Psi_4 = 0$ when $Y_4 = 0$, $0 < x_4 < \delta_0/\alpha_0$. Then the terminal condition (4.10) on u_4 becomes

$$u_4^2 + (\pi^2/4\delta_0^2)(\Psi_4^2 + 2V_w \bar{x}_5 \Psi_4) = 0 \quad \text{at} \quad x_4 = \delta_0/\alpha_0 \quad (0 < Y_4 < \delta_0). \tag{5.1}$$

All streamlines in this subregion pass through the part $0 < Y_4 < \delta_0$ of the line $x_4 = \delta_0/\alpha_0$ and hence on any such streamline

$$u_4^2 = -2P_4(x_4) + 2P_4(\delta_0/\alpha_0) - (\pi^2/4\delta_0^2)(\Psi_4^2 + 2V_w \bar{x}_5 \Psi_4). \tag{5.2}$$

Further the streamline $\Psi_4 = \tilde{\Psi}_4$ leaves IV_3 by entering IV_2 at a value of x_4 such that

$$\tilde{\Psi}_4 = -\frac{T_w^*}{T_\infty^*} C^{\frac{1}{2}} \int_0^{x_4} v_E(x_4)\, dx_4. \tag{5.3}$$

Thus
$$\alpha_0 x_4 = \int_0^{\tilde{\Psi}_4} \frac{d\Psi_4}{u_4} = \frac{2\delta_0}{\pi}\left[\sin^{-1}\frac{V_w \bar{x}_5}{Q} - \sin^{-1}\frac{V_w \bar{x}_5 + \tilde{\Psi}_4}{Q}\right], \tag{5.4}$$

where
$$Q^2 = (8\delta_0^2/\pi^2)\,[P_4(\delta_0/\alpha_0) - P_4(x_4)] + (V_w \bar{x}_5)^2. \tag{5.5}$$

The set of equations (5.3)–(5.5) enables us to find P_4 as a function of x_4 by making use of table 1, which gives the variation of v_E with x_4. When $x_4 = \delta_0/\alpha_0$, the relevant value of $\tilde{\Psi}_4$ is $-v_4 \bar{x}_5$, corresponding to the streamline which enters IV_2 directly from V_3. The constant $P_4(\delta_0/\alpha_0)$ is indeterminate but as explained earlier this is not significant at the present stage of the asymptotic expansion. When x_4 is small, $|\tilde{\Psi}_4| \ll 1$ and so we have

$$P_4 \approx -\tilde{\Psi}_4^2/2\alpha_0^2 x_4^2, \quad u_4 \approx \tilde{\Psi}_4/\alpha_0 x_4. \tag{5.6}$$

Further
$$\tilde{\Psi}_4 \simeq -(T_w^*/T_\infty^*)\, C^{\frac{1}{2}} C_0 \lambda^{\frac{1}{2}} x_4^{\frac{3}{2}} \tag{5.7}$$

using (3.8) and (3.9), whereupon (3.9) reduces to a form agreeing with (2.7), thus establishing the consistency of the match between IV_2 and III_4.

At $x_4 = \delta_0/\alpha_0$, the slip velocity is zero from (4.10) but, at smaller values of x_4, it increases under the action of the favourable pressure gradient and eventually becomes singular at $x_4 = 0$, as indicated by (5.6) and (5.7). A sub-boundary layer IV_4, of thickness $O(\epsilon^3 x_s^*)$, is therefore needed to reduce the slip velocity to zero. It generates a normal velocity $O(\epsilon^5 U_\infty^*)$ at the bottom of IV_3 which may be regarded as a small perturbation of the flow already found for that subregion. Near $x_4 = \delta_0/\alpha_0$

$$P_4\left(\frac{\delta_0}{\alpha_0}\right) - P_4(x_4) \simeq \left[\frac{\alpha_0 \pi}{2\delta_0} V_w \bar{x}_5 + \tilde{\Psi}_4'\left(\frac{\delta_0}{\alpha_0}\right)\right]^2 \frac{\pi^2}{8\delta_0^2}\left(\frac{\delta_0}{\alpha_0} - x_4\right)^2, \tag{5.8}$$

showing that the sub-boundary layer begins with a stagnation-point similarity solution. Near $x_4 = 0$ it also takes on a similarity form, which is identical with that for III_5. In this form (Stewartson & Williams 1973) the vorticity decays algebraically as the similarity variable tends to infinity but this presents no matching difficulties with the solution in IV_3. For (Brown & Stewartson 1965) this form of decay only occurs at $x_4 = 0+$, and at all values of $x_4 > 0$ the decay is exponential. Strictly the sub-boundary layer begins in region V but as explained earlier the slip velocities are then $O(\epsilon^3 U_\infty^*)$ and negligible to leading order.

6. The structure of regions VI and VII

Region VI lies above the permeable part of the plate and its principal features have already been discussed in P. The top subregion (VI_1) is a continuation of IV_1 and V_1, the flow in it is inviscid and the only new property is that its lower boundary bends round under the influence of a favourable pressure gradient until it is eventually parallel to the plate at the termination of the blow. This pressure gradient is needed to drive the injected fluid downstream. The next

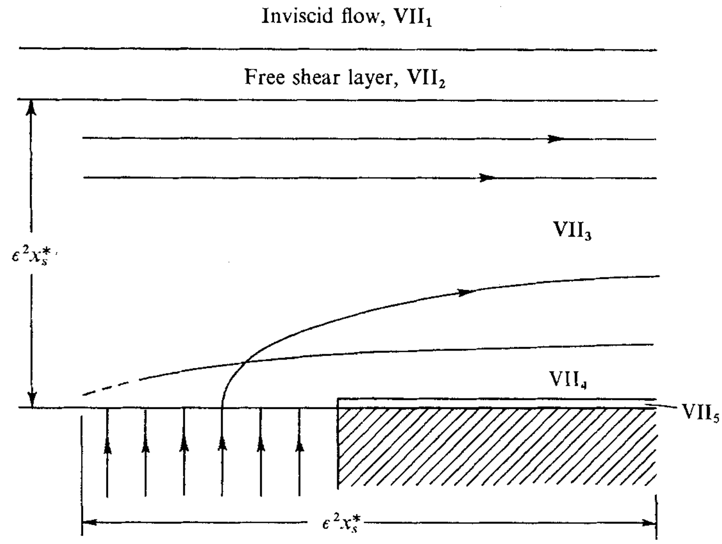

FIGURE 5. The subregions of VII (the termination of blowing).

subregion down (VI_2) is the continuation of the shear layer in IV_2 and V_2 and has the same properties. In particular the evolution of v_E is independent of the blow. The next, and lowest, subregion (VI_3) contains the blown fluid; the x^* component of velocity is $O(\epsilon U_\infty^*)$ and its thickness is $O(\epsilon^2 x_s^*)$. The theory of P describes the flow in VI_3 to leading order and the presence of VI_2 causes perturbations in u^* of order $\epsilon^2 U_\infty^*$. Finally there is no need for a sub-boundary layer equivalent to IV_4.

At the end $x^* = x_1^*$ of VI, the pressure is required to satisfy the condition $p^* - p_\infty^* = O(\epsilon^3 \rho_\infty^* U_\infty^{*2})$ and this fixes the value of x_s^*, the principal parameter of the problem not specified *a priori*. We wish to demonstrate here that the discontinuity in the pressure gradient at $x^* = x_1^*$, which is implied by this requirement, can be smoothed out by a transition region in the neighbourhood of $x^* = x_1^*$.

The various subregions of VII are shown in figure 5. Of these, VII_1 and VII_2 need no further comment. In order to set up an appropriate structure for VII_3, we write

$$\left.\begin{aligned} x^* &= x_1^* + \epsilon^2 x_s^* x_7, \quad y^* = \epsilon^2 x_s^* Y_7, \quad u^* = \epsilon U_\infty^* U_7(Y_7) + \epsilon^3 U_\infty^* u_7(x_7, Y_7), \\ v^* &= \epsilon^3 U_\infty^* v_7(x_7, Y_7), \quad p^* = p_\infty^* + \epsilon^4 \rho_w^* U_\infty^{*2} p_7(x_7, Y_7). \end{aligned}\right\} \tag{6.1}$$

Here $\epsilon U_\infty^* U_7$ is the limit of the solution for u^* in VI_3 at $x = x_1^*$, and is known from P. The pressure in VI_3 is also known from P and we can write

$$p^* = p_\infty^* + \epsilon^2 \rho_w^* U_\infty^{*2} \left[P_{61} \frac{x^* - x_1^*}{x_s^*} + \tfrac{1}{2} P_{62} \left(\frac{x^* - x_1^*}{x_s^*} \right)^2 + \dots \right] \tag{6.2}$$

for its form when $(x^* - x_1^*)/x_s^*$ is small, where P_{61} and P_{62} can be supposed known constants.

Then, in VII_3, the governing equations reduce to

$$\left.\begin{aligned} & U_7\frac{\partial u_7}{\partial x_7}+v_7\frac{dU_7}{dY_7} = -\frac{\partial p_7}{\partial x_7}, \quad U_7\frac{\partial v_7}{\partial x_7} = -\frac{\partial p_7}{\partial Y_7}, \\ & u_7 = \partial\Psi_7/\partial Y_7, \quad v_7 = -\partial\Psi_7/\partial x_7, \end{aligned}\right\} \tag{6.3}$$

together with the boundary conditions

$$v_7 = \left\{\begin{matrix} V_w, 0 & \text{if} \quad Y_7 = 0, & x_7 \lessgtr 0, \\ 0 & \text{if} \quad Y_7 = \delta_1, & \end{matrix}\right\} \tag{6.4}$$

where $\epsilon^2\delta_1 x_s^*$ is the thickness of VI_3 at $x^* = x_1^*$. On eliminating p_7 from (6.3) we obtain

$$\frac{\partial^2\Psi_7}{\partial x_7^2}+\frac{\partial^2\Psi_7}{\delta Y_7^2} = \frac{1}{U_7}\frac{d^2U_7}{dY_7^2}\Psi_7. \tag{6.5}$$

The coefficient of Ψ_7 on the right-hand side of (6.5) is finite at $Y_7 = 0$, even though $U_7(0) = 0$, and is equal to P_{62}/V_w^2. As $x_7 \to -\infty$, $u_7 - x_7\overline{U}_7'(Y_7)$ remains finite and $v_7 \to -\overline{U}_7(Y_7)$, where

$$\overline{U}_7(Y_7) = P_{61}U_7(Y_7)\int_{Y_7}^{\delta_1}\frac{dY_7}{U_7^2}; \tag{6.6}$$

as $x_7 \to +\infty$, $\Psi_7 \to 0$. Notice that $\overline{U}_7'(0) = 0$.

The solution of (6.5) requires knowledge of the behaviour of U_7 and in general can only be completed in numerical terms. However a representative form for U_7 is $\bar{A}\sin(\pi Y_7/2\delta_1)$, where $\bar{A}$ is a constant; this form is proportional to the initial profile for u^* in VI_3. The solution of (6.5) now follows easily because the right-hand side reduces to $-\pi^2\Psi_7/4\delta_1^2$. We write

$$\Psi_7 = \left\{\begin{matrix} x_7\overline{U}_7(Y_7)+\Psi_- & \text{if} \quad x_7 < 0, \\ \Psi_+ & \text{if} \quad x_7 > 0. \end{matrix}\right\} \tag{6.7}$$

Then
$$\Psi_\pm = \sum_{n=1}^{\infty} A_n\frac{\pi n Y_7}{\delta_1}\exp\left(\mp\frac{\pi x_7}{\delta_1}(n^2-\tfrac{1}{4})^{\frac{1}{2}}\right),$$

where
$$2\sum_{n=1}^{\infty} A_n(n^2-\tfrac{1}{4})^{\frac{1}{2}}\sin\frac{n\pi Y_7}{\delta_1} = -\frac{\delta_1}{\pi}\overline{U}_7(Y_7); \tag{6.8}$$

if we use the representative form for U_7,

$$A_n(n^2-\tfrac{1}{4})^{\frac{3}{2}} = \frac{2n\delta_1^2}{\pi^3\bar{A}}P_{61} = \frac{n\delta_1 V_w}{\pi^2}. \tag{6.9}$$

According to this solution the tangential component of velocity is non-zero at $Y_7 = 0$, for

$$u_7(x_7, 0) = F_7(x_7) = \frac{\pi}{\delta_1}\sum_{n=1}^{\infty} nA_n e^{\mp\lambda_n x_7} \quad \text{in} \quad x_7 \gtrless 0, \quad \text{where} \quad \lambda_n = (\pi/\delta_1)(n^2-\tfrac{1}{4})^{\frac{1}{2}}. \tag{6.10}$$

This slip velocity is reduced to zero in sublayers which take on different forms in $x_7 < 0$ and in $x_7 > 0$. Considering first $x_7 < 0$, we find that a sublayer VII_4

of thickness $O(\epsilon^3 x_s^*)$ is needed and that it is also inviscid in character and arises because the expansion (6.1) is non-uniform near $Y_7 = 0$. We write $\epsilon Z_7 = Y_7$,

$$\left.\begin{aligned} u^*/U_\infty^* &= \epsilon^2 U_7'(0)\, Z_7 + \epsilon^3 \hat{U}_7(x_7, Z_7) + \ldots, \\ v^*/U_\infty^* &= \epsilon^3 V_w + \epsilon^4 \hat{V}_7(x_7, Z_7) + \ldots, \\ p^* &= p_\infty^* + \epsilon^4 \rho_w^* U_\infty^{*2} p_7(x_7, 0) + \epsilon^5 \rho_\infty^* U_\infty^{*2} \hat{p}_7(x_7, Z_7) + \ldots. \end{aligned}\right\} \tag{6.11}$$

On substituting these expansions into the full equations of motion, we obtain

$$\left.\begin{aligned} &\frac{\partial \hat{U}_7}{\partial x_7} + \frac{\partial \hat{V}_7}{\partial Z_7} = 0, \quad \frac{\partial \hat{p}_7}{\partial Z_7} = 0, \\ &U_7'(0)\left(Z_7 \frac{\partial \hat{U}_7}{\partial x_7} + \hat{V}_7\right) + V_w \frac{\partial \hat{U}_7}{\partial Z_7} = -\frac{\partial \hat{p}_7}{\partial x_7}. \end{aligned}\right\} \tag{6.12}$$

The solution of (6.12) must satisfy the boundary conditions $\hat{U}_7(x_7, 0) = \hat{V}_7(x_7, 0)$, $\hat{U}_7(x_7, \infty) = F_7(x_7)$ and $\hat{U}_7(-\infty, Z_7) = 0$. It is

$$\hat{U}_7 = \frac{\pi}{\delta_1} \sum_{n=1}^{\infty} n A_n\, e^{\lambda_n x_7} \operatorname{erf}\left\{\left(\frac{\lambda_n U_7'(0)}{2V_w}\right)^{\frac{1}{2}} Z_7\right\} \quad (x_7 < 0). \tag{6.13}$$

Associated with this solution is a pressure gradient $\hat{p}_7$ and an effective displacement thickness which serve to induce velocities $O(\epsilon^4 U_\infty^*)$ in VII_3.

A further non-uniformity occurs in the flow field in the neighbourhood of $x_7 = 0$ whose structure cannot be resolved by boundary-layer considerations alone, but we shall not investigate its properties here. In $x_7 > 0$ the sublayer needed to reduce the slip velocity (6.10) to zero takes on a double structure. The continuation of VII_4 is controlled by (6.12) except that V_w is now zero and hence

$$\left.\begin{aligned} \hat{U}_7 &= F_7(x_7) - \frac{\pi}{\delta_1} \sum_{n=1}^{\infty} n A_n \operatorname{erfc}\left[\left(\frac{\lambda_n U_7'(0)}{2V_w}\right)^{\frac{1}{2}} Z_7\right], \\ \hat{V}_7 &= -Z_7 F_7'(x_7), \quad \hat{p}_7 = \text{constant} \quad \text{in} \quad x_7 > 0. \end{aligned}\right\} \tag{6.14}$$

According to (6.9) and (6.14)

$$\hat{U}_7 - \frac{2V_w}{\pi} \log Z_7 \to F_7(x_7) + \text{constant}, \tag{6.15}$$

as $Z_7 \to 0$ and a further sublayer VII_5 is therefore needed to reduce $\hat{U}_7$ to zero at the plate. Viscous effects are significant in this sublayer and its thickness is $O(\epsilon^{\frac{11}{3}} x_s^*)$. The governing equation is the boundary-layer equation, linearized about the uniform shear flow $\epsilon^2 U_\infty^* U_7'(0)\, Z_7$ and, notwithstanding the singular behaviour of $\hat{U}_7$ as $Z_7 \to 0$ in (6.15), its solution presents no formal difficulties.

7. Conclusion

With the discussion of VII, the study of the most significant regions of the flow field has been completed. The subsequent history of the flow field, downstream of VII, is complicated but, with one proviso, has no bearing on the leading terms of the asymptotic expansion of the solution in the regions upstream of x_1^*, since the pressure variation there is too small. Let us suppose, to fix matters, that the flow fields in $y^* > 0$ and $y^* < 0$ are mirror images, so that, downstream of the trailing edge at $x^* = L$, the appropriate conditions at $y^* = 0$ are $v^* = \partial u^*/\partial y^* = 0$. The change in the boundary conditions at $(L, 0)$ necessitates the study of a new set of subregions in its neighbourhood, analogous to the triple deck discussed by Messiter (1970) and Stewartson (1969). Further downstream the shear layers in $y^* > 0$ and $y^* < 0$ continue to grow in width, entraining fluid from the slowly moving flow between them. In addition the fluid on the centre-line of this slowly moving subregion is being accelerated by viscous action, as in the Goldstein (1930) near-wake theory.

In order to discuss the final structure of the flow, however, we have to order the limits $\epsilon \to 0$, $x^*/L \to \infty$, and this is the proviso mentioned earlier. In the present paper we shall take them in the order stated, or equivalently, take the view that our interest in the flow field ceases when x^* is a large but finite multiple of L. Consideration of other limit orderings is more difficult and is deferred for the present.

The main conclusions of this paper are that the model of the flow field assumed by P for strong plate-injection is justified and that it is possible to set up a formal expansion of the solution in powers of ϵ, even though it is first necessary to divide the flow field into a large number of overlapping subregions. In particular we have shown how the reversed flow set up in the triple deck at separation is fed by the injected fluid and how the flow field readjusts itself at the termination of the blow, so permitting the pressure gradient rapidly to fall to zero there. In view of the consistency of the matches between the various subregions, we venture to claim that the structure found can form the basis of a strict mathematical conjecture about the nature of the solution of the Navier–Stokes equations for this problem, as $\epsilon \to 0$.

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